Math 247A Lecture 21 Notes

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February 26, 2020

1 Estimates on the Littlewood-Paley Square Function and the Fractional Product Rule

1.1 Estimates on the Littlewood-Paley square function

Theorem 1.1 (Littlewood-Paley square function estimate). *Let $f \in \mathcal{S}(\mathbb{R}^d)$ and define the square function*

$$S(f) = \sqrt{\sum |f_N|^2}.$$

Then

$$\|S(f)\|_p \sim_p \|f\|_p \qquad \forall 1 < p < \infty.$$

Proof. Let $\{X_N\}_{N \in 2^{\mathbb{Z}}}$ be iid random variables with $X_n = \pm 1$ with equal probability, and define the random variable $m_X(\xi) = \sum X_n \psi_N(\xi)$. Last time, we showed that m_X is a Mikhlin multiplier, uniformly in the choice of X_N. This holds even if we replace ψ be another $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ function. Now

$$m_X^\vee * f = \sum_{N \in 2^{\mathbb{Z}}} X_N f_N.$$

By Kinchine's inequality,

$$\mathbb{E}[|m_X^\vee * f|^2]^{1/p} \sim \sqrt{\sum |f_N|^2} \sim_p S(f).$$

Now

$$\begin{aligned} \|S(f)\|_p^p &\sim \int \mathbb{E}[|m_X^\vee * f|^p(x)]\, dx \\ &\sim \mathbb{E}\Bigg[\underbrace{\int |m_X^\vee * f|^p(x)\, dx}_{\|m_X^\vee * f\|_p^p} \Bigg] \\ &\lesssim \mathbb{E}[\|f\|_p^p] \end{aligned}$$

$$\lesssim \|f\|_p^p.$$

Again, note that this holds for any $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ function in place of ψ.

To prove the reverse inequality, we argue by duality and use the generality under which we proved the first inequality. We say

$$\begin{aligned} \|f\|_p &= \sup_{\|g\|_{p'}=1} \langle f, g\rangle \\ &= \sup_{\|g\|_{p'}=1} \left\langle \sum P_N f, g \right\rangle \end{aligned}$$

Since $\widetilde{P}_N P_N = P_N$ and $\widetilde{P}_n$ is self-adjoint,

$$\begin{aligned} &= \sup_{\|g\|_{p'}=1} \sum_{N\in 2^{\mathbb{Z}}} \langle P_N f, \widetilde{P}_N g\rangle \\ &\le \sup_{\|g\|_{p'}=1} \int \sqrt{\sum_N |P_N f|^2} \sqrt{\sum_N |\widetilde{P}_N g|^2}\, dx \end{aligned}$$

Using Hölder,

$$\le \|S(f)|_p \sup_{\|g\|_{p'}=1} \left\| \sqrt{\sum_N |\widetilde{P}_N g|^2} \right\|_{p'}.$$

Replacing ψ by $\widetilde{\psi}(\xi) = \psi(2\xi) + \psi(\xi) + \psi(\xi/2) \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ in the previous argument, we get

$$\left\| \sqrt{\sum_N |\widetilde{P}_N g|^2} \right\|_{p'} \lesssim \|g\|_{p'} \lesssim 1. \qquad \square$$

Corollary 1.1. *Fix $1 < p < \infty$. Then*

1. *Whenever $s > -d$ and $f \in \mathcal{S}(\mathbb{R}^d)$ (or $s \in \mathbb{R}$ and $\widehat{f} \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$),*

$$\||\nabla|^s f\|_p \sim_p \left\| \sqrt{\sum N^{2s} |f_N|^2} \right\|_p.$$

2. *For $s > 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\||\nabla|^s f\|_p \sim_p \left\| \sqrt{\sum N^{2s} |f_{\ge N}|^2} \right\|_p.$$

Proof.

1. Let's show that $\left\| \sqrt{\sum N^{2s}|f_N|^2} \right\|_p \lesssim \| |\nabla|^s f\|_p$. We have

$$\begin{aligned}\sum N^{2s}|f_N|^2 &= \sum N^{2s}||\nabla|^{-s}|\nabla|^s f_N|^2\\ &= \sum |N^s|\nabla|^{-s}P_N(|\nabla|^s f)|^2\end{aligned}$$

Replacing ψ by $\chi(\xi) = \frac{1}{(2\pi|\xi|)^s}\psi(\xi) \in C_c^\infty(\mathbb{R}^d\backslash\{0\}$ and ψ_N by $\chi_N(\xi) = \left(\frac{N}{2\pi|\xi|}\right)^s \psi_N(\xi)$, we get

$$\left\| \sqrt{\sum |N^s|\nabla|^{-s}P_N(|\nabla|^s f)|^2} \right\|_p \lesssim_p \||\nabla|^s f\|_p.$$

To prove the reverse inequality, we argue by duality:

$$\begin{aligned}\||\nabla|^s f\|_p &= \sup_{\|g\|_{p'}=1} \langle |\nabla|^s f, g\rangle\\ &= \sup_{\|g\|_{p'}=1} \sum_N \left\langle |\nabla|^s f_N, \widetilde{P}_N g\right\rangle\\ &= \sup_{\|g\|_{p'}=1} \sum_N \left\langle N^s f_N, N^{-s}|\nabla|^s \widetilde{P}_N g\right\rangle\end{aligned}$$

Recall that $\mathcal{F} = \{h \in \mathcal{S}(\mathbb{R}^d) : \widehat{h}$ vanishes in a nbhd of $0\}$ is dense in $L^{p'}$. So we can always take g to be in this family. So

$$\begin{aligned}\||\nabla|^s f\|_p &\leq \sup_{\substack{g\in\mathcal{F}\\ \|g\|_{p'}=1}} \int \sqrt{\sum N^{2s}|f_N|^2}\sqrt{\sum N^{-2s}||\nabla|^s\widetilde{P}_n g|^2}\,dx\\ &\leq \left\| \sqrt{\sum N^{2s}|f_N|^2} \right\|_p \sup_{\|g\|_{p'}=1} \left\| \sqrt{\sum N^{-2s}||\nabla|^s \widetilde{P}_N g|^2} \right\|_p.\end{aligned}$$

Replacing ψ by

$$\chi(\xi) = (2\pi|\xi|)^s\widetilde{\psi}(\xi), \qquad \chi_n(\xi) = \left(\frac{2\pi|\xi|}{N}\right)^s \widetilde{\psi}_N(\xi),$$

we get

$$\left\| \sqrt{\sum N^{-2s}||\nabla|^s\widetilde{P}_N g|^2} \right\|_{p'} \lesssim \|g\|_{p'} \lesssim 1.$$

2. We claim that $\sum N^{2s}|f_{\geq N}|^2 \sim \sum N^{2s}|f_N|^2$. We have

$$\sum_N N^{2s}|f_{\geq N}|^2 = \sum_N N^{2s}\left(\sum_{N_1\geq N} f_{N_1}\right)\left(\overline{\sum_{N_2\geq N} f_{N_2}}\right)$$

By paying a factor of 2, we can assume $N_1 \leq N_2$.

$$\begin{aligned}
&\leq 2 \sum_{N \leq N_1 \leq N_2} N^{2s} |f_{N_1}| \cdot |f_{N_2}| \\
&\lesssim \sum_{N_1 \leq N_2} N_1^{2s} |f_{N_1}||f_{N_2}| \\
&\lesssim \sum_{N_1 \leq N_2} \left(\frac{N_1}{N_2}\right)^s (N_1^s |f_{N_1}|) (N_2^s |f_{N_2}|)
\end{aligned}$$

By Cauchy-Schwarz,

$$\lesssim \sum_N N^{2s} |f_N|^2.$$

On the other hand,

$$|f_N| = |f_{\geq N} - f_{\geq 2N}| \leq |f_{\geq N} f_{\geq 2N}|.$$

So

$$\begin{aligned}
\sum_N N^{2s} |f_N|^2 &\lesssim \sum_N N^{2s} |f_{\geq n}|^2 + 2^{-2s} \sum_N (2N)^{2s} |f_{\geq 2N}|^2 \\
&\lesssim \sum_N N^{2s} |f_{\geq N}|^2.
\end{aligned}$$

□

1.2 The fractional product rule

Theorem 1.2 (Fractional product rule, Christ-Weinstein, 1991)**.** *Fix* $s > 0$ *and* $1 < p, p_1, p_2, q_1, q_2 < \infty,$. *Then*

$$\||\nabla|^s (fg)\|_p \lesssim \||\nabla|^s f\|_{p_1} \|g\|_{p_2} + \|f\|_{q_1} + \||\nabla|^s g\|_{q_2}.$$

whenever $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

Remark 1.1. p_2 and q_1 are allowed to be ∞.

We really should only be proving this for $0 < s < 1$, since for integers, we can just use the regular product rule and then look at the fractional part.

Proof. We have

$$\||\nabla|^s (fg)\|_p \sim \left\| \sqrt{\sum_N N^{2s} |P_N (fg)|^2} \right\|_p.$$

We write $fg = f_{\geq N/4}g + f_{>N/4}g_{\geq N/4} + f_{<N/4}g_{<N/4}$, so

$$P_N(fg) = P_N(f_{\geq N/4}g) + P_N(f_{>N/4}g_{\geq N/4}) + \underbrace{P_N(f_{<N/4}g_{<N/4})}_{\to 0}.$$

This gives

$$\begin{aligned}|P_N(fg)| &\lesssim M(f_{\geq N/4}g) + M(f_{\leq N/4}g_{\geq N/4}) \\ &\lesssim M(f_{\geq N/4}g) + M((Mf)g_{\geq N/4}).\end{aligned}$$

So we get

$$\sum N^{2s}|P_N(fg)|^2 \lesssim \sum |M((N^s f_{\geq N/4})g)|^2 + \sum |((Mf)\cdot N^s g_{\geq N/4})|^2,$$

which gives

$$\sqrt{\sum N^{2s}|P_N(fg)|^2} \lesssim \sqrt{\sum |M((N^s f_{\geq N/4})g)|^2} + \sqrt{\sum |((Mf)\cdot N^s g_{\geq N/4})|^2}.$$

So we get

$$\||\nabla|^s(fg)\|_p \lesssim \left\|\sqrt{N^{2s}|f_{\geq} N/4|^2}g\right\|_p + \left\|Mf\sqrt{N^{2s}|g_{\geq} N/4|^2}\right\|_p$$

By the corollary,

$$\lesssim \||\nabla|^s f\|_{p_1}\|g\|_{p_2} + \|f\|_{q_1}\||\nabla|^s g\|_{q_2}. \qquad \square$$